CS 61B Spring 2025

Asymptotics II

Exam-Level 05: February 24, 2025

1 Finish the Runtimes

Below we see the standard nested for loop, but with missing pieces!

```
for (int i = 1; i < ______; i = ______) {
    for (int j = 1; j < ______; j = ______) {
        System.out.println("Circle is the best TA");
    }
}
```

For each part below, **some** of the blanks will be filled in, and a desired runtime will be given. Fill in the remaining blanks to achieve the desired runtime! There may be more than one correct answer.

Hint: You may find `Math.pow` helpful.

(a) Desired runtime: $\Theta(N^2)$

```
for (int i = 1; i < N; i = i + 1) {
    for (int j = 1; j < i; j = ______) {
        System.out.println("This is one is low key hard");
    }
}
```

(b) Desired runtime: $\Theta(\log(N))$

```
for (int i = 1; i < N; i = i * 2) {
    for (int j = 1; j < ______; j = j * 2) {
        System.out.println("This is one is mid key hard");
    }
}
```

(c) Desired runtime: $\Theta(2^N)$. $\frac{2^N}{N}$ is a valid answer, could you think of another?

```
for (int i = 1; i < N; i = i + 1) {
    for (int j = 1; j < ______; j = j + 1) {
        System.out.println("This is one is high key hard");
    }
}
```

(d) Desired runtime: $\Theta(N^3)$

```
for (int i = 1; i < ______; i = i * 2) {
    for (int j = 1; j < N * N; j = ______) {
        System.out.println("yikes");
    }
}
```

2 Asymptotics is Fun!

(a) Using the function `g` defined below, what is the runtime of the following function calls? Write each answer in terms of `N`. Feel free to draw out the recursion tree if it helps.

```
void g(int N, int x) {
    if (N == 0) {
        return;
    }
    for (int i = 1; i <= x; i++) {
        g(N - 1, i);
    }
}
```

`g(N, 1)`: Θ(____)

`g(N, 2)`: Θ(____)

(b) Suppose we change line 6 to `g(N - 1, x)` and change the stopping condition in the for loop to `i <= f(x)` where `f` returns a random number between 1 and `x`, inclusive. For the following function calls, find the tightest Ω and big O bounds. Feel free to draw out the recursion tree if it helps.

```
void g(int N, int x) {
    if (N == 0) {
        return;
    }
    for (int i = 1; i <= f(x); i++) {
        g(N - 1, x);
    }
}
```

`g(N, 2)`: Ω(____), O(____)

`g(N, N)`: Ω(____), O(____)

3 Asymptotics Proofs

As a reminder, the formal definitions of Ω, Θ, and O are provided below:

Let f, g be real-valued functions. Then:

$f(x) \in \Theta(g(x))$ if there exists $a, b, N_0 > 0$ such that for all $N > N_0$, $|ag(N)| \leq |f(N)| \leq |bg(N)|$.

$f(x) \in O(g(x))$ if there exists $b, N_0 > 0$ such that for all $N > N_0$, $|f(N)| \leq |bg(N)|$.

$f(x) \in \Omega(g(x))$ if there exists $a, N_0 > 0$ such that for all $N > N_0$, $|ag(N)| \leq |f(N)|$.

Informally, we say that $f(x) \in O(g(x))$ approximately means that $f(x) \leq g(x)$, and similarly, $f(x) \in \Theta(g(x))$ means $f(x) = g(x)$ and $f(x) \in \Omega(g(x))$ means $f(x) \geq g(x)$. This problem will explore why we can make this informal statement, by showing that the O relation shares many properties with the $\leq$ relation.

For this problem, let f, g, and h be real-valued functions, and let x, y, and z be real numbers. You won't be expected to write full proofs on exams, but this thinking style will be helpful on exams and especially in later classes.

(a) If $x \leq y$, then $y \geq x$. Show that if $f(x) \in O(g(x))$, then $g(x) \in \Omega(f(x))$

(b) If $x \leq y$ and $y \leq x$, then $x = y$. Show that if $f(x) \in O(g(x))$ and $g(x) \in O(f(x))$, then $f(x) \in \Theta(g(x))$

(c) For any real number, $x \leq x$. Show that for any function, $f(x) \in O(f(x))$.

(d) If $x \leq y$ and $y \leq z$, then $x \leq z$. Show that if $f(x) \in O(g(x))$ and $g(x) \in O(h(x))$, then $f(x) \in O(h(x))$

(e) For any pair of real numbers x and y, either $x < y$, $x = y$, or $x > y$. Show that this is NOT a property of O; that is, find functions f and g such that $f(x) \notin O(g(x))$ and $g(x) \notin O(f(x))$.